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APPLICATIONS OF INDUCED RESULTANTS TO POLYNOMIAL MAPS

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ABSTRACT

Let $F(X,Y)$ be a two dimensional polynomial map over C. We show how to use the notion of induced resultants in order to give short and elementary proofs to the following three theorems:

1. *If the Jacobian of F is a non-zero constant, then the image of F contains all of C^2 except for a finite set.*
2. *If F is invertible, then the inverse map is determined by the free terms of the induced resultants.*
3. *If F is invertible, then the degree of F equals the degree of its inverse.*

1. Introduction

Let $P(X,Y), Q(X,Y) \in \mathbb{C}[X,Y]$ and let α, β be two indeterminates. We will denote by

$$\text{resultant}(P(X,Y)-\alpha, Q(X,Y)-\beta, X), \quad \text{resultant}(P(X,Y)-\alpha, Q(X,Y)-\beta, Y)$$

the two induced resultants of the pair. As shown in [2] the structure of the variety of the asymptotic values of the polynomial map $F=(P,Q)$ is closely related to those resultants.

In this note we give three applications of the induced resultants:

THEOREM 1: *If the Jacobian of F is a non-zero constant, then the image of F contains all of $\mathbb{C}^2$ except for a finite set.*

THEOREM 2: *If F is invertible, then the inverse map is determined by the free terms of the induced resultants (as polynomials in α, β).*

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THEOREM 3: *If F is invertible, then the degree of F equals the degree of its inverse.*

The three theorems are known to experts in this form or another, but the proofs given here are simple and readable.

2. Picard's Little Theorem for polynomial maps

Let $P(X,Y), Q(X,Y) \in \mathbb{C}[X,Y]$ be a Jacobian pair. Then they cannot share a common level curve in the sense that there are two constants $\alpha, \beta \in C$ such that for any X there exists a $Y(X)$ such that

$$P(X,Y(X)) = \alpha \quad \text{and} \quad Q(X,Y(X)) = \beta.$$

Hence there cannot exist a pair (α_0, β_0) such that

$$\text{resultant}(P(X,Y) - \alpha_0, Q(X,Y) - \beta_0, Y) \equiv 0.$$

This proves the following

FACT: Let $P(X,Y), Q(X,Y) \in C[X,Y]$ be a Jacobian pair. Let α and β be indeterminates. Let

$$\text{resultant}(P(X,Y) - \alpha, Q(X,Y) - \beta, Y) = R_N(\alpha,\beta)X^N + \cdots + R_0(\alpha,\beta)$$

where $R_N(\alpha,\beta)$ is the highest non-vanishing (identically) coefficient of

$$\text{resultant}(P(X,Y) - \alpha, Q(X,Y) - \beta, Y).$$

Then $R_j(\alpha,\beta)$, $0 \leq j \leq N$, cannot share a zero. In particular, the greatest common divisor of $R_N(\alpha,\beta), \ldots, R_0(\alpha,\beta)$ is 1.

THEOREM 1: *Let*

$$P(X,Y), Q(X,Y) \in \mathbb{C}[X,Y] \quad \textit{and} \quad F(X,Y) = (P(X,Y), Q(X,Y)).$$

Then there are finitely many polynomials $R_N(\alpha,\beta), \ldots, R_1(\alpha,\beta), R_0(\alpha,\beta) \in \mathbb{C}[\alpha,\beta]$ such that

$$\begin{aligned}\mathbb{C}^2 - F(\mathbb{C}^2) &= \{(a,b) \in \mathbb{C}^2 \mid R_N(a,b) = \cdots = R_1(a,b) = 0, R_0(a,b) \neq 0\} \\ &= V_{\mathbb{C}}(R_N, \ldots, R_1) \cap (\mathbb{C}^2 - V_{\mathbb{C}}(R_0).\end{aligned}$$

Moreover, if P, Q is a Jacobian pair then $\mathbb{C}^2 \setminus F(\mathbb{C}^2)$ is a finite set.

Remark 1: This theorem is the analog of Picard's Little Theorem for analytic functions. Needless to mention is that the last portion of this theorem is well known.

Proof: Let α, β be two new indeterminates and let us form

$$R(X) = \text{resultant}(P(X,Y) - \alpha, Q(X,Y) - \beta, Y) \in \mathbb{C}[X, \alpha, \beta].$$

Suppose that we have the following representation of $R(X)$ as a polynomial in X:

$$R(X) = R_N(\alpha,\beta)X^N + \cdots + R_1(\alpha,\beta)X + R_0(\alpha,\beta)$$

where $R_N(\alpha,\beta), \ldots, R_0(\alpha,\beta) \in C[\alpha,\beta]$. Then

$(a,b) \in C^2 - F(C^2)$

$\iff$ there is no solution in (X,Y) to $P(X,Y) - a = Q(X,Y) - b = 0$

$\iff$ for any X_0, $R(X_0) \neq 0$ where $(\alpha,\beta) \leftarrow (a,b)$

$\iff$ $R(X) \in \mathbb{C}^*$ where $(\alpha,\beta) \leftarrow (a,b)$

$\iff$ $R(X) = R_N(a,b)X^N + \cdots + R_0(a,b) = R_0(a,b) \neq 0$.

This proves the first part of the theorem. Now for the second part: Suppose that P, Q form a Jacobian pair. Then the last assertion is equivalent (by the FACT) to the following:

$$R_N(a,b) = \cdots, R_1(a,b) = 0.$$

Moreover, by the Bezout Theorem

$C^2 - F(C^2)$ is infinite

$\iff$ $L(\alpha,\beta) = (R_N(\alpha,\beta), \ldots, R_1(\alpha,\beta)) \in C[\alpha,\beta] - C^*$

$\iff$ $L(P(X_0,Y_0), Q(X_0,Y_0)) \neq 0$ for all (X_0,Y_0)

$\iff$ $L(P(X,Y), Q(X,Y)) = c \in \mathbb{C}^*$

$\iff$ $\partial(P,Q)/\partial(X,Y) \equiv 0$. ∎

3. Invertible morphisms

Let $P(X,Y), Q(X,Y) \in \mathbb{C}[X,Y]$ such that $F(X,Y) = (P(X,Y), Q(X,Y))$ is an invertible morphism. Let

$$\text{resultant}(P(X,Y) - \alpha, Q(X,Y) - \beta, Y) = R_N X^N + R_{N-1}(\alpha,\beta)X^{N-1} + \cdots + R_0(\alpha,\beta).$$

Since $F(X,Y)$ is a morphism it follows that $P(X,Y), Q(X,Y)$ is a Jacobian pair, so by the results in the previous section, $N \geq 1$. Also, since $F(X,Y)$

cannot have asymptotic values it follows that $R_N \in \mathbb{C}^*$ (see [2]). Clearly, $R_{N-1}(\alpha,\beta),\dots,R_0(\alpha,\beta) \in \mathbb{C}[\alpha,\beta]$. Given any (α_0,β_0) and any X_0 which is a zero of

$$R_N X^N + \cdots + R_0(\alpha_0,\beta_0), \tag{1}$$

there exists a Y_0 such that $F(X_0,Y_0) = (\alpha_0,\beta_0)$. Since $F(X,Y)$ is injective it follows that all the zeros of (1) must coincide. This proves that

$$\begin{aligned} R_N X^N + \cdots + R_0(\alpha_0,\beta_0) &= R_N\left(X^N + \frac{R_{N-1}(\alpha,\beta)}{R_N}X^{N-1} + \cdots + \frac{R_0(\alpha,\beta)}{R_N}\right) \\ &= R_N\left(X + \left(\frac{R_0(\alpha,\beta)}{R_N}\right)^{1/N}\right)^N. \end{aligned} \tag{2}$$

In particular, we obtain the following $N+1$ relations:

$$R_j(\alpha,\beta)/R_N = \binom{N}{j}\left(\frac{R_0(\alpha,\beta)}{R_N}\right)^{(N-j)/N}, \quad 0 \le j \le N. \tag{3}$$

In particular, for $j = N-1$ we get

$$\left(\frac{R_0(\alpha,\beta)}{R_N}\right)^{1/N} = \frac{R_{N-1}(\alpha,\beta)}{(NR_N)} \in \mathbb{C}[\alpha,\beta]. \tag{4}$$

A similar argument applies to

$$\operatorname{resultant}(P(X,Y)-\alpha, Q(X,Y)-\beta, X)$$

We have just proved the following

FACT: Let $P(X,Y), Q(X,Y) \in \mathbb{C}[X,Y]$ such that

$$F(X,Y) = (P(X,Y),Q(X,Y))$$

is an invertible morphism. Then there exist two positive integers N and M such that

$$\operatorname{resultant}(P(X,Y)-\alpha, Q(X,Y)-\beta, Y) = R_N(X + r_0(\alpha,\beta))^N$$

and

$$\operatorname{resultant}(P(X,Y)-\alpha, Q(X,Y)-\beta, X) = S_M(X + s_0(\alpha,\beta))^M$$

where $R_N, S_M \in \mathbb{C}^*$ and $r_0(\alpha,\beta), s_0(\alpha,\beta) \in \mathbb{C}[\alpha,\beta]$.

MORE ANALYSIS:

Let us assume that we have the following standard representations:

$$(5)\qquad \begin{aligned} P(X,Y) &= a_n(X)Y^n + \cdots + a_0(X), \quad \deg_Y P(X,Y) = n,\\ Q(X,Y) &= b_m(X)Y^m + \cdots + b_0(X), \quad \deg_Y Q(X,Y) = m. \end{aligned}$$

Then by Sylvester's formula we get

$$(6)\qquad \operatorname{resultant}(P(X,Y)-\alpha, Q(X,Y)-\beta, Y)$$

$$= \begin{vmatrix} a_n(X) & \dots & 0 & b_m(X) & \dots & 0 \\ \vdots & \dots & a_n(X) & \vdots & \dots & b_m(X) \\ a_0(X)-\alpha & \dots & \vdots & b_0(X)-\beta & \dots & \vdots \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ 0 & \dots & a_0(X)-\alpha & 0 & \dots & b_0(X)-\beta \end{vmatrix}.$$

So in particular we have

$$(7)\qquad R_0(\alpha,\beta) = \begin{vmatrix} a_n(0) & \dots & 0 & b_m(0) & \dots & 0 \\ \vdots & \dots & a_n(0) & \vdots & \dots & b_m(0) \\ a_0(0)-\alpha & \dots & \vdots & b_0(0)-\beta & \dots & \vdots \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ 0 & \dots & a_0(0)-\alpha & 0 & \dots & b_0(0)-\beta \end{vmatrix}.$$

Hence $\deg R_0(\alpha,\beta) \le \max(n,m)$ or, equivalently,

$$(8)\qquad \deg R_0(\alpha,\beta) \le \max(\deg_Y P(X,Y), \deg_Y Q(X,Y)) = \deg_Y F(X,Y).$$

By (2) we obtain the following conclusion: If $P(X,Y)=\alpha$ and $Q(X,Y)=\beta$, then $X(\alpha,\beta) = -(R_0(\alpha,\beta)/R_N)^{1/N}$. By (4), $X(\alpha,\beta) \in \mathbb{C}[\alpha,\beta]$, and by (8), $\deg X(\alpha,\beta) \le \deg_Y F(X,Y)$. Similarly, if

$$\begin{aligned} &\operatorname{resultant}(P(X,Y)-\alpha, Q(X,Y)-\beta, X)\\ &\qquad = S_M Y^M + S_{M-1}(\alpha,\beta)Y^{M-1} + \cdots + S_0(\alpha,\beta) \end{aligned}$$

then we have

$$(9)\qquad S_M Y^M + \cdots + S_0(\alpha,\beta) = S_M \left(Y + \left(\frac{S_0(\alpha,\beta)}{S_M}\right)^{1/M}\right)^M,$$

(10) $$\frac{S_j(\alpha,\beta)}{S_M} = \binom{M}{j}\left(\frac{S_0(\alpha,\beta)}{S_M}\right)^{(M-j)/M}, \quad 0 \le j \le M,$$

(11) $$\left(\frac{S_0(\alpha,\beta)}{S_M}\right)^{1/M} = \frac{S_{M-1}(\alpha,\beta)}{(MS_M)} \in \mathbb{C}[\alpha,\beta],$$

(12) $$\deg S_0(\alpha,\beta) \le \max(\deg_X P(X,Y), \deg_X Q(X,Y)) = \deg_X F(X,Y).$$

So by (9) we obtain the following conclusion: If $P(X,Y)=\alpha$ and $Q(X,Y)=\beta$, then $Y(\alpha,\beta) = -(S_0(\alpha,\beta)/S_M)^{1/M}$. By (11), $Y(\alpha,\beta) \in \mathbb{C}[\alpha,\beta]$; and by (12), $\deg Y(\alpha,\beta) \le \deg_X F(X,Y)$.

Remark 2: It is possible to show that $N = M = 1$.

We now can conclude our second and third theorems:

THEOREM 2: *Let $P(X,Y), Q(X,Y) \in \mathbb{C}[X,Y]$ and suppose that $F(X,Y) = (P(X,Y), Q(X,Y))$ is an invertible morphism. Let us assume that we have the following standard representations:*

$$\begin{aligned} P(X,Y) &= a_n(X)Y^n + \cdots + a_0(X), \quad \deg_Y P(X,Y) = n,\\ &= A_N(Y)X^N + \cdots + A_0(Y), \quad \deg_X P(X,Y) = N;\\ Q(X,Y) &= b_m(X)Y^m + \cdots + b_0(X), \quad \deg_Y Q(X,Y) = m,\\ &= B_M(Y)X^M + \cdots + B_0(Y), \quad \deg_X Q(X,Y) = M. \end{aligned}$$

Let $F^{-1}(\alpha,\beta) = (X(\alpha,\beta), Y(\alpha,\beta))$. Then there are $R_1, S_1 \in \mathbb{C}^*$ so that we have the following formulas for the inverse map:

$$-R_1X(\alpha,\beta) = \begin{vmatrix} a_n(0) & \dots & 0 & b_m(0) & \dots & 0\\ \vdots & \dots & a_n(0) & \vdots & \dots & b_m(0)\\ a_0(0)-\alpha & \dots & \vdots & b_0(0)-\beta & \dots & \vdots\\ \vdots & \dots & \vdots & \vdots & \dots & \vdots\\ 0 & \dots & a_0(0)-\alpha & 0 & \dots & b_0(0)-\beta \end{vmatrix}$$

and

$$-S_1Y(\alpha,\beta) = \begin{vmatrix} A_N(0) & \dots & 0 & B_M(0) & \dots & 0\\ \vdots & \dots & A_N(0) & \vdots & \dots & B_M(0)\\ A_0(0)-\alpha & \dots & \vdots & B_0(0)-\beta & \dots & \vdots\\ \vdots & \dots & \vdots & \vdots & \dots & \vdots\\ 0 & \dots & A_0(0)-\alpha & 0 & \dots & B_0(0)-\beta \end{vmatrix}.$$

Proof: This follows by the discussion after (8) and by (7). ∎

THEOREM 3: *Let* $P(X,Y), Q(X,Y) \in C[X,Y]$ *such that*

$$F(X,Y) = (P(X,Y), Q(X,Y))$$

is an invertible morphism. Let us denote $F^{-1}(\alpha,\beta) = (X(\alpha,\beta), Y(\alpha,\beta))$. *Then*

$$\begin{aligned}
\deg X(\alpha,\beta) &\le \max(\deg_Y P(X,Y), \deg_Y Q(X,Y)) = \deg_Y F(X,Y),\\
\deg Y(\alpha,\beta) &\le \max(\deg_X P(X,Y), \deg_X Q(X,Y)) = \deg_X F(X,Y),\\
\deg P(X,Y) &\le \max(\deg_\beta X(\alpha,\beta), \deg_\beta Y(\alpha,\beta)) = \deg_\beta F^{-1}(\alpha,\beta),\\
\deg Q(X,Y) &\le \max(\deg_\alpha X(\alpha,\beta), \deg_\alpha Y(\alpha,\beta)) = \deg_\alpha F^{-1}(\alpha,\beta),
\end{aligned}$$

$$\deg F(X,Y) = \deg F^{-1}(\alpha,\beta).$$

Proof: The first two inequalities were proved in equations (8) and (12) and the discussion that followed. The second two inequalities follow from the first two by changing the roles of F and F^{-1}. The fifth equality is a conclusion of the previous four inequalities. ∎

Remark 3: The fifth equality appears in [1] on page 292. It is proved there for any dimension (with the appropriate modification). It was communicated to the authors of [1] by Ofer Gaber, who attributed it to an unrecalled colloquium lecturer at Harvard. The authors mention that John Tyrrell (King's College, University of London) has indicated that this equality was well known to classical geometers.

References

[1] H. Bass, E. Connell and D. Wright, *The Jacobian conjecture: reduction of degree and formal expansion of the inverse,* Bulletin of the American Mathematical Society **7** (1982), 287–330.

[2] R. Peretz, *The variety of the asymptotic values of a complex polynomial map,* Preprint, Ben-Gurion University of the Negev, Beer-Sheva, Israel, 1997.